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Large deviations and nontrivial exponents in coarsening systems

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Abstract. We investigate the statistics of the mean magnetization, of its large deviations and *persistent large deviations* in simple coarsening systems. In particular we consider more specifically the case of the diffusion equation, of the Ising chain at zero temperature and of the two-dimensional voter model. For the diffusion equation, at large times, the mean magnetization has a limit law, which is studied analytically using the independent interval approximation. The probability of persistent large deviations, defined as the probability that the mean magnetization was, for all previous times, greater than some level x, decays algebraically at large times, with an exponent $\theta(x)$ continuously varying with x. When $x = 1$, $\theta(1)$ is the usual persistence exponent. Similar behaviour is found for the Glauber–Ising chain at zero temperature. For the two-dimensional voter model, large deviations of the mean magnetization are algebraic, while the probability of persistent large deviations seem to behave as the usual persistence probability.

1. Introduction

To date most studies of persistence in simple nonequilibrium systems have focused on the behaviour of the persistence probability at large times and on the computation of the related persistence exponent [1–28]. The aim of this paper is to broaden the scope of these former studies by investigating the statistics of more general persistent events. We shall see that consideration of these events leads, in particular, to the introduction of new nontrivial exponents.

A simple definition of persistence may be given as follows. Let a time-dependent random variable $\sigma(t)$ take only two values ± 1, with some dynamical rule. Think for instance of $\sigma(t)$ as being the spin at a particular site in a dynamical Ising model. The persistence of this random variable up to time t corresponds to the most extreme situation where it never changed sign. In other terms the spin spent all of its time in only one of the two possible phases. Note that, by its very definition, this event is nonlocal in time. The probability of this event, or persistence probability, for most of the systems mentioned above, decreases algebraically in time, with nontrivial exponents. The surprise of finding new nontrivial exponents in the dynamics of nonequilibrium systems motivated, to a large extent, the interest for the subject.

In this paper we investigate the statistics of *large deviations* and of *persistent large deviations* for simple coarsening systems. Both are natural generalizations of the concept of persistence. We apply this study to the case of the diffusion equation, the one-dimensional Glauber–Ising chain at zero temperature, and the two-dimensional voter model.

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Let us define the mean 'magnetization' at time t of the random process $\sigma(t)$, or 'spin' for short, as

$$M(t) = \frac{1}{t}\int_0^t \mathrm{d}u\, \sigma(u) \tag{1.1}$$

which is such that $-1 \leqslant M(t) \leqslant 1$. The quantities considered in this work are the following.

We first define the distribution of the mean magnetization by

$$P(t, x) = P(M(t) \geqslant x). \tag{1.2}$$

For $x > 0$ this quantity measures the chance for the mean magnetization to deviate from its average†. In the regime of large times, and for x much larger than the width of the probability density function of M, $P(t, x)$ is referred to as the probability of large deviations.

We then define

$$R(t, x) = P(M(u) \geqslant x, \forall u \leqslant t). \tag{1.3}$$

This quantity will be hereafter referred to as the probability of *persistent large deviations*.

Persistence, as defined above, corresponds to the largest deviation such that $M(t) = 1$ (assuming for instance that $\sigma = 1$ initially). The persistence probability therefore reads

$$R(t) = P(\sigma(u) = 1, \forall u \leqslant t) = P(t, 1) = R(t, 1) \tag{1.4}$$

thus appearing as a limiting case of the two previous probabilities.

We find, for the models considered in this work, the following results in the long time regime.

(i) For the diffusion equation, $M(t)$ has a limit law, i.e. $P(t, x)$ converges to a limit distribution $P_\infty(x)$ when $t \to \infty$. Using the independent interval approximation, the moments of this limit distribution are computed analytically. Their behaviour at high orders, or the singular behaviour of $P_\infty(x)$ for $x \to 1$, are related to the persistence exponent θ.

The probability $R(t, x)$ is found numerically to behave as $t^{-\theta(x)}$, with an exponent $\theta(x)$ varying continuously from 0 for $x = -1$, to θ, the usual persistence exponent, for $x = 1$. (See section 3.)

(ii) For the one-dimensional Ising model at zero temperature, similar behaviour is found, namely $P(t, x) \to P_\infty(x)$ and $R(t, x) \sim t^{-\theta(x)}$. (See section 4.)

(iii) For the two-dimensional voter model, numerical simulations suggest that, in the regime of large deviations, $P(t, x)$ behaves as $t^{-\tilde{\theta}(x)}$, with an exponent continuously varying with x ($x > 0$). They also seem to indicate that $R(t, x)$ behaves as $\exp[-J(x)(\ln t)^2]$. (See section 5.)

We devote the next section to further considerations on large deviations and persistent large deviations. A general discussion and generalizations shall be given in section 6.

2. Large deviations and persistent large deviations

Let us comment on definitions (1.2) and (1.3).

First it is obvious that large deviations reflect a persistence property of the process. Think for instance of an event such that $M(t)$ takes a value very close to 1, corresponding to a very large deviation. This is even more true of a *persistent* large deviation which is a more constrained event.

† Hereafter we shall only consider cases with zero average magnetization, i.e. such that the average $\langle\sigma(t)\rangle$ of the spin (and therefore of $M(t)$) over histories is zero.

Let us define the *occupation time* of the phases $(+)$ or $(-)$, i.e. the time spent in the $\sigma = \pm$ phase, by

$$T_{\pm} = \int_0^t \mathrm{d}u \, \frac{1 \pm \sigma(u)}{2} = t \left(\frac{1 \pm M(t)}{2} \right). \tag{2.1}$$

In other words, the mean magnetization $M(t)$ of a generic spin gives a measure of the fraction of time that this spin spent in one of the two phases. Therefore, while a large deviation only requires that a generic spin was, up to time t, *most of the time* in the same phase, in such a way that $M(t) \geqslant x$, a persistent large deviation constrains the spin to fulfil this condition *at all previous times*. Finally persistence corresponds for the spin to staying *always* in the same phase, hence $R(t) = P(T_+ = t)$. (Assuming that initially $\sigma = -1$ would lead to the definition $R(t) = P(T_- = t)$.)

Let us now illustrate the previous definitions on the very simple example of a symmetric random walk on a one-dimensional lattice. We denote by $\sigma(t)$ the step made by the walker at the discrete time t, where $\sigma = \pm 1$ with probability $\frac{1}{2}$. Starting from the origin at time 0, the position of the walker at time t is given by $\sum_1^t \sigma(u)$. Its average position is equal to 0. The quantity $M(t)$ introduced above represents the mean speed of the walker.

The law of the position of the walker is well known. At large times it is normally distributed around its mean, with a variance proportional to t. As a consequence, the density of M is peaked around $x = 0$, with a variance decreasing as $1/t$. The probability of a large deviation, giving a measure of the chance for the walker to reach a position far away from the origin, is exponentially small, and is given by (see appendix A)

$$P(t, x) \sim \mathrm{e}^{-tI(x)} \qquad (x > 0, t \gg 1) \tag{2.2}$$

where $I(x) = (\frac{1}{2})[(1 + x)\ln(1 + x) + (1 - x)\ln(1 - x)]$ is an entropy function. In other words, the law of large numbers holds, the mean $M(t)$ converging to its average $\langle M \rangle = 0$, when $t \to \infty$. The limit law of M is a delta peak centred at $x = 0$, and $P(t, x) \to P_\infty(x) = H(-x)$, where $H(x)$ is the Heaviside function.

The persistent large deviation $M(u) \geqslant x, \forall u \leqslant t$ corresponds to a situation where the walker always had a mean speed larger than x, i.e. stayed to the right of the position xt, between 0 and t. If $x > 0$, $R(t, x)$ behaves at large times in a similar fashion as in equation (2.2). If $x < 0$, $R(t, x)$ has a limit $R_\infty(x)$ when $t \to \infty$ which is a decreasing function of x, with a discontinuity at every rational value of x [29]. In the marginal case $x = 0$, it is easy to show that $R(t, x) \approx 1/\sqrt{\pi t}$, for t large.

Finally persistence corresponds, for the walker, to always stepping in the same direction. The persistence probability is

$$R(t) = \mathrm{e}^{-t \ln 2}. \tag{2.3}$$

Note that $I(1) = \ln 2$.

By analogy with the case of the random walk, we set, for the models studied in this work,

$$P(t, x) \sim \mathrm{e}^{-a(t)I(x)} \qquad (x > 0, t \gg 1) \tag{2.4}$$

which defines a function $a(t)$ characteristic of the temporal behaviour of large deviations, and an entropy function $I(x)$, keeping the same notation as above (see appendix A). In a similar fashion, setting

$$R(t, x) \sim \mathrm{e}^{-b(t)J(x)} \qquad (t \gg 1) \tag{2.5}$$

defines a function $b(t)$ characteristic of the temporal behaviour of the persistent large deviations.

Table 1. Summary of results for the functions $a(t)$, $b(t)$, $I(x)$, $J(x)$ (see equations (2.4) and (2.5)).

	$a(t)$	$b(t)$	$I(x \to 1)$	$J(x \to 1)$
Random walk	t	t	$\ln 2$	$\ln 2$
Diffusion equation / One-dimensional Ising	$\mathcal{O}(1)$	$\ln t$	∞	θ
Two-dimensional voter	$\ln t$	$(\ln t)^2$	∞	constant

The time dependence of $a(t)$ and $b(t)$ for the models studied in this work is summarized in table 1 and shall be discussed in the following sections.

Let us mention some mathematical references relevant for this work. Occupation times have been studied for Markov processes [30], and for several infinite particle systems [31, 32]. Large deviations for occupation times were studied in [33–36]. These references shall be commented upon in the course of the paper. We are not aware of previous references on persistent large deviations.

3. The diffusion equation

3.1. The independent interval approximation and the persistence exponent

First we introduce definitions, and recall results, which shall be needed in the next section. Consider the equation

$$\partial_t \phi(\boldsymbol{x}, t) = \nabla^2 \phi(\boldsymbol{x}, t) \tag{3.1}$$

where $\phi(\boldsymbol{x}, 0)$ is Gaussian, with zero mean. Here $\boldsymbol{x}$ denotes a point in d-dimensional space. The changes of sign, or zero crossings, of the field ϕ at a given space point, occur at times $t_1, t_2, \ldots, t_n$, starting from some time origin, or in the variable $\tau = \ln t$, at times $\tau_1, \tau_2, \ldots, \tau_n$.

Define, for a given space point $\boldsymbol{x}$, the process $\Phi(t) = \phi(\boldsymbol{x}, t)/\sqrt{\langle \phi(\boldsymbol{x}, t)^2 \rangle}$. This process is Gaussian and stationary in the time variable τ, i.e. its two-time correlation function $\langle \Phi(\tau_1)\Phi(\tau_2)\rangle = [\cosh(\tau_2 - \tau_1)/2]^{-d/2}$ only depends on the difference $|\tau_2 - \tau_1|$ [11, 12]. As a consequence, the autocorrelation of the process $\sigma = \operatorname{sign}(\Phi)$ reads [11, 12]

$$A(\tau) = \langle \sigma(0)\sigma(\tau)\rangle = \frac{2}{\pi} \sin^{-1} \frac{1}{(\cosh \tau/2)^{d/2}} \tag{3.2}$$

with the Laplace transform

$$\hat{A}(s) = \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-s\tau} A(\tau) = \frac{1}{s}\left(1 - \frac{d}{2\pi} I_d(s)\right) \tag{3.3}$$

where

$$I_d(s) = \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-s\tau} \frac{\tanh \tau/2}{\sqrt{(\cosh \tau/2)^d - 1}}. \tag{3.4}$$

Let us denote by $l_n = \tau_n - \tau_{n-1}$ the intervals between zero crossings in the τ variable. Considering the intervals as independent reduces the zero crossing process to a renewal process, entirely described, in the stationary regime, by $f(l)$, the probability density function of intervals. For such a process the probability $p_n(\tau)$ of having exactly n zero crossings up to time τ reads, in the Laplace space,

$$\begin{aligned} \hat{p}_n(s) &= \frac{(1-\hat{f})^2}{s^2\langle l\rangle}\hat{f}^{n-1}(s) \qquad (n>0) \\ \hat{p}_0(s) &= \frac{1}{s} - \frac{1-\hat{f}}{s^2\langle l\rangle}. \end{aligned} \tag{3.5}$$

Noting that $A(\tau) = \sum_{n=0}^{\infty}(-)^n p_n(\tau)$, leads to the following relation between $\hat{f}$ and $\hat{A}$ [11, 12]

$$\hat{f}(s) = \frac{1-\langle l\rangle s(1-s\hat{A})/2}{1+\langle l\rangle s(1-s\hat{A})/2} = \frac{1-s\sqrt{d/2}I_d(s)}{1+s\sqrt{d/2}I_d(s)} \tag{3.6}$$

with $\langle l\rangle = \pi\sqrt{8/d}$.

The persistence probability $R(t)$ is the probability that the field ϕ at a given space point did not change sign up to time t. Equivalently it is the probability that $\sigma(\tau)$ did not flip up to time τ, i.e. the probability of no zero crossing $p_0(\tau)$. At large times it behaves as $t^{-\theta}$, or as $\mathrm{e}^{-\theta\tau}$. As a consequence, at large l, $f(l) \sim \mathrm{e}^{-\theta l}$, the persistence exponent appearing as the rightmost pole of $\hat{f}(s)$, $s = -\theta$ [11, 12].

For example, in one dimension:

$$\begin{aligned} I_1(s) &= \sqrt{2}\int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-s\tau}\frac{\cosh\tau/4}{\cosh\tau/2} = \sqrt{2}(\beta(s+\tfrac{1}{4}) + \beta(s+\tfrac{3}{4})) \\ &= \sqrt{2}\sum_{p=0}^{\infty}(-)^p\left(\frac{1}{s+\frac{1}{4}+p} + \frac{1}{s+\frac{3}{4}+p}\right) \end{aligned} \tag{3.7}$$

with $I_1(0) = 2\pi$. The function $\beta(x)$ is related to $\psi(x)$, the logarithmic derivative of the gamma function, by

$$\beta(x) = \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right]. \tag{3.8}$$

In two dimensions:

$$\begin{aligned} I_2(s) &= \int_0^\infty \mathrm{d}\tau\,\frac{\mathrm{e}^{-s\tau}}{\cosh\tau/2} = 2\beta(s+\tfrac{1}{2}) \\ &= 2\sum_{p=0}^{\infty}(-)^p\frac{1}{s+\frac{1}{2}+p} \end{aligned} \tag{3.9}$$

with $I_2(0) = \pi$. The largest zero of $1 + s\sqrt{d/2}I_d(s)$ is found to be at $s = -\theta$, with $\theta = 0.120\,327\,978\,84\ldots$, for $d = 1$ and $\theta = 0.186\,221\,071\,297\ldots$, for $d = 2$.

3.2. Statistics of the mean magnetization

In this section our concern is the determination of the distribution of the random variable $M(t)$, at large times. We denote by t, or by τ in the logarithmic scale, the observation time and by λ the 'backward recurrence time', i.e. the length of time measured backwards from τ to the last crossing event before τ: $\lambda = \tau - \tau_n$. The probability distribution of λ in Laplace space reads, in the stationary regime†,

$$\hat{q}(s) = \frac{1 - \hat{f}(s)}{s\langle l \rangle}. \tag{3.10}$$

We have, assuming that $\sigma(t) = 1$,

$$M(t) = \frac{1}{t}\int_0^t \mathrm{d}u\, \sigma(u) = \frac{1}{t}(t - t_n - (t_n - t_{n-1}) + \cdots) = 1 - 2\xi \tag{3.11}$$

where

$$\xi = \frac{t_n}{t} - \frac{t_{n-1}}{t} + \cdots = \mathrm{e}^{-\lambda}(1 - \mathrm{e}^{-l_n} + \mathrm{e}^{-l_n - l_{n-1}} - \cdots) = \mathrm{e}^{-\lambda} X_n. \tag{3.12}$$

Assuming that $\sigma(t) = -1$ leads to $M(t) = 2\xi - 1$. Note that $\xi = T_{\mp}/t$, according to the sign of $\sigma(t)$, i.e. ξ is the fraction of time spent in the 'wrong' phase (cf equation (2.1)). The random variable $X_n = 1 - \mathrm{e}^{-l_n} + \mathrm{e}^{-l_n - l_{n-1}} - \cdots$ obeys the recursion relation $X_n = 1 - \mathrm{e}^{-l_n} X_{n-1}$. It is therefore recognized as a Kesten variable [37–40].

To summarize, in the limit $t \to \infty$, the three equations

$$M = \pm(1 - 2\xi) \tag{3.13a}$$

$$\xi = \mathrm{e}^{-\lambda} X \tag{3.13b}$$

$$X = 1 - \mathrm{e}^{-l} X \tag{3.13c}$$

contain the relevant information for the determination of $f_M(x) = -\mathrm{d}P_\infty(x)/\mathrm{d}x$, the distribution of M. Equation (3.13c) should be understood as an equality in distribution.

The determination of the probability density of the Kesten variable X (and hence of $f_M(x)$) for any given distribution of intervals $f(l)$ is known in general as a hard problem, and does not seem feasible in this case. However, from this set of equations we are able to extract the following information.

(i) We perform a local analysis of $f_M(x)$, for $x \to 1$.

(ii) We compute the moments $\langle M^k \rangle$ of $f_M(x)$.

(iii) We solve the set of equations (3.13a–c) for the case of an exponential distribution of intervals, $f(l) = \theta \mathrm{e}^{-\theta l}$, proportional to the tail of the true distribution given, in the Laplace space, by equation (3.6).

Let us analyse the local behaviour of $f_M(x)$ in the persistence region $x \to 1$ (a similar analysis would hold in the limit $x \to -1$). Then $\xi \to 0$, i.e. $\lambda \to \infty$, with X finite. These conditions define the persistence region, where $s \approx -\theta$. Therefore $f(l) \approx a\mathrm{e}^{-\theta l}$ and $q(\lambda) \approx a\mathrm{e}^{-\theta l}/\langle l \rangle \theta$, where a is the residue of $\hat{f}(s)$ for $s = -\theta$. Consider the Mellin transform of the law of ξ, $\langle \xi^s \rangle$. From (3.13b) one obtains

$$\langle \xi^s \rangle = \hat{q}(s)\langle X^s \rangle \tag{3.14}$$

where $\hat{q}(s)$ is the Laplace transform (3.10). In this regime, one has $\langle \xi^s \rangle \approx b/(s + \theta)$, with b given by

$$b = \frac{a}{\langle l \rangle \theta}\langle X^{-\theta} \rangle. \tag{3.15}$$

By inversion of the Mellin transform equation (3.14) one obtains the behaviour of the distribution of ξ, hence that of M in the persistence region $x \to 1$. One finds

$$f_M(x) \approx 2^{-\theta-1} b (1 - x)^{\theta - 1} \qquad (x \approx 1). \tag{3.16}$$

† Let us note that the age of the system considered in [21] is just equal to $t - t_n$. It is related to the scaling variable $t_n/t = \mathrm{e}^{-\lambda}$, the distribution of which is known in the case considered here.

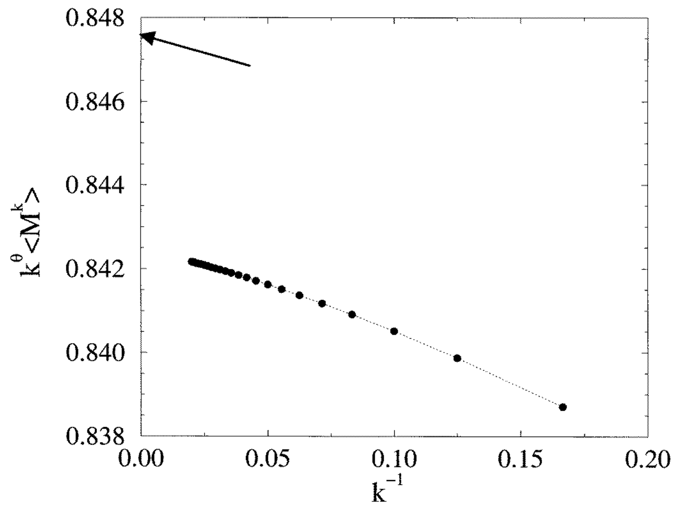

Figure 1. Plot of $k^{\theta}\langle M^k\rangle$ ($k = 2, 4, \ldots, 50$) for the two-dimensional diffusion equation in the independent interval approximation, versus k^{-1}, showing the approach to the limit amplitude 0.8424. The arrow represents the value of the limit amplitude 0.8476 for the beta law (see (3.21)).

As a consequence, for large k one has

$$\langle M^k\rangle \approx 2^{-\theta} b\Gamma(\theta) k^{-\theta}. \tag{3.17}$$

Note that the determination of b requires that of $\langle X^{-\theta}\rangle$, which is unknown. A numerical estimate of the amplitude b can nevertheless be given, as follows. Using the method given in appendix B, we computed the numerical values of the first 50 moments of M in one and two dimensions, in the independent interval approximation. By extrapolating these results we find $2^{-\theta} b\Gamma(\theta) \approx 0.870$ for $d = 1$, and $2^{-\theta} b\Gamma(\theta) \approx 0.8424$ for $d = 2$. In figure 1 a plot of $k^{-\theta}\langle M^k\rangle$ versus $1/k$, for $d = 2$, is given, showing the approach to the limit amplitude.

We are naturally led to compare the distribution $f_M(x)$ to a beta law on $(-1, 1)$, with the same singular behaviour in the region $x \approx 1$,

$$f^{\mathrm{Beta}}(x) = B^{-1}(\tfrac{1}{2}, \theta)(1 - x^2)^{\theta-1} \tag{3.18}$$

with

$$B(\tfrac{1}{2}, \theta) = \frac{\Gamma(\frac{1}{2})\Gamma(\theta)}{\Gamma(\frac{1}{2} + \theta)}. \tag{3.19}$$

The even moments of this law are given by

$$\mu_k = \frac{B((k+1)/2, \theta)}{B(\frac{1}{2}, \theta)}. \tag{3.20}$$

The odd moments are zero, by construction. At large orders,

$$\mu_k \approx 2^{\theta} \frac{\Gamma(\frac{1}{2} + \theta)}{\Gamma(\frac{1}{2})} k^{-\theta} \qquad (k \gg 1). \tag{3.21}$$

The amplitude $2^{\theta}\Gamma(\frac{1}{2} + \theta)/\Gamma(\frac{1}{2})$ is equal to 0.8858 for $d = 1$, and to 0.8476 for $d = 2$. Defining the ratio of local amplitudes A by $f_M \approx A f^{\mathrm{Beta}}$, for $x \approx 1$, yields

$$A = \lim_{k\to\infty} \frac{\langle M^k\rangle}{\mu_k} = bB(\theta, \theta + 1) \tag{3.22}$$

Table 2. Values of the moments $\langle M^k \rangle$ for the diffusion equation, computed in the independent interval approximation, compared with the moments of the beta law (3.18).

	$\langle M^2 \rangle$	μ_2	$\langle M^4 \rangle$	μ_4	$\langle M^6 \rangle$	μ_6
One-dimensional	0.7996	0.8060	0.7383	0.7462	0.7035	0.7119
Two-dimensional	0.7268	0.7286	0.6459	0.6482	0.6008	0.6032

using (3.17) and (3.21). Hence using the numerical values given above, we find $A \approx 0.982$ for $d = 1$, and $A \approx 0.9938$ for $d = 2$. Table 2 gives the values of the first moments $\langle M^k \rangle$, compared with the moments of the beta law (3.18).

The resemblance of $f_M(x)$ to the beta law (3.18) is enhanced by the fact that the solution of (3.13*a*–*c*) for an exponential distribution of intervals $f(l) = \theta \mathrm{e}^{-\theta l}$, proportional to the tail of the true distribution $f(l)$ given, in the Laplace space, by equation (3.6), is precisely given by (3.18) (see appendix B). This demonstrates the dominance of the tail of $f(l)$ for the determination of $f_M(x)$.

We also computed the probability distribution of the mean magnetization obtained by numerical integration of equation (3.1), for $d = 1$. This distribution is also found to be very close to f^{Beta}.

In summary, the mean magnetization $M(t)$ has a limit distribution when $t \to \infty$. In other words there is no law of large numbers for the random process $\sigma(t)$, and absence of ergodicity. This distribution is found to be extremely close to the beta distribution (3.18). As long as $\theta < 1$, the density $f_M(x)$ diverges for $x \to \pm 1$. Therefore the most probable values of M are near -1 and 1, while the average $\langle M \rangle = 0$. The probability of a 'large deviation', i.e. in the regime $t \to \infty$, x close to 1, is large. Finally, the divergence when $x \to 1$ of the function $I(x)$ defined in (2.4), signals the crossover to the persistence regime.

3.3. Persistent large deviations of the mean magnetization

We performed numerical simulations of the diffusion equation, equation (3.1) in one dimension, for a system size equal to 10^6, starting from a random initial condition. At large times one observes an algebraic decay of the probability of persistent large deviations $R(t, x)$ of the form

$$R(t, x) \sim t^{-\theta(x)} \qquad (-1 \leqslant x \leqslant 1) \tag{3.23}$$

which corresponds to the behaviour $b(t) \sim \ln t$ for the function defined in (2.5). The exponent $\theta(x)$ is to be identified to $J(x)$ defined in equation (2.5).

Figure 2 gives a plot of the probability of persistent large deviations $R(t, x)$ for $x = -0.8$. The usual persistence probability $R(t)$ is also plotted, for comparison. The third curve corresponds to $R(t, x, y)$ defined at the end of section 6 (see the comment there).

Figure 3 shows a plot of the exponent $\theta(x)$ for $-1 \leqslant x \leqslant 1$. The exponent varies continuously from 0, for $x = -1$, to the value of the usual persistence exponent $\theta \approx 0.121$, for $x = 1$. (We recall that the value of $\theta \approx 0.121$ obtained by numerical integration of (3.1) is slightly larger than the value of the exponent obtained by the independent interval approximation [11, 12].) We shall comment further on these results in section 6.

Let us finally mention that similar results to those presented in figures 2 and 3 are found in two dimensions.

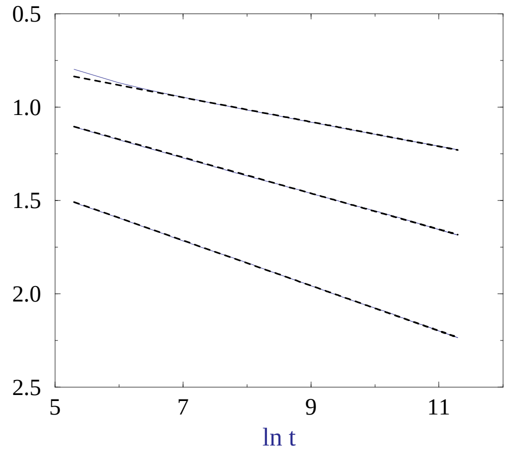

Figure 2. Persistence probability $R(t)$, probability of persistent large deviations $R(t, x)$, and $R(t, x, y)$ (see section 6), for the one-dimensional diffusion equation. (System size $= 10^6$.) From bottom to top: $R(t)$ (slope $= -0.121$), $R(t, -0.8)$ (slope $= -0.096$), $R(t, -0.8, -0.8)$ (slope $= -0.065$). In broken curves: regression lines.

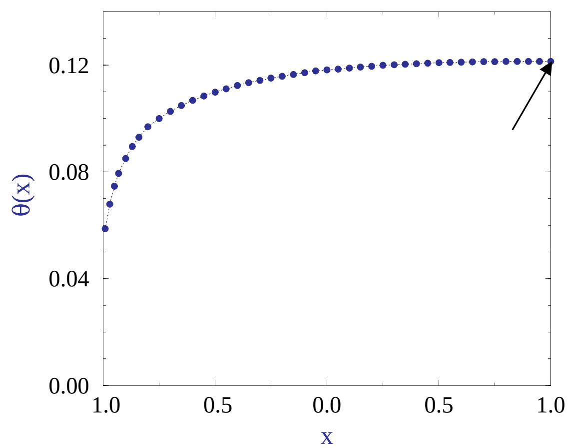

Figure 3. Exponent $\theta(x)$ for the one-dimensional diffusion equation. The arrow represents the usual persistence exponent $\theta \approx 0.121$.

4. The Glauber–Ising chain

We studied the Ising chain at zero temperature with the following dynamics [41]. On each site of the one-dimensional lattice, values of the spin $\sigma = \pm$ are initially distributed randomly. At each timestep a site is picked at random. The spin on this site takes the value of one of its neighbours, chosen at random.

We performed numerical simulations on a system of size $L = 10^6$. As for the case of the diffusion equation, $P(t, x)$ has a limit distribution when $t \to \infty$. This distribution is very close to a beta law corresponding to the persistence exponent $\theta = \frac{3}{8}$ [8]. The analytical study of $P(t, x)$ will be given elsewhere. In particular it is easy to understand why this distribution converges to a limit when $t \to \infty$. For instance $\langle M^2 \rangle = \hat{A}(1)$, where $\hat{A}(s)$ is

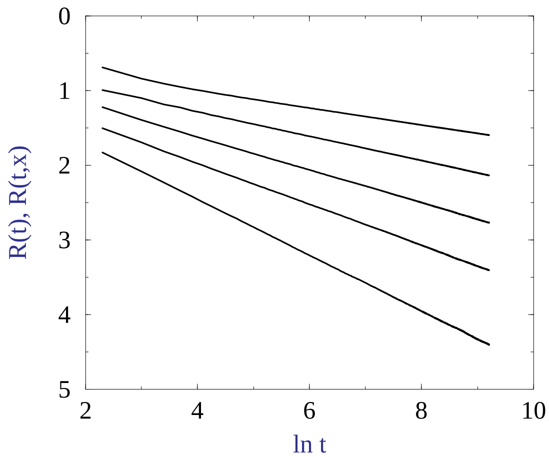

Figure 4. Probability of persistent large deviations $R(t, x)$ for the one-dimensional Ising model. (System size $= 10^6$.) From bottom to top: $R(t)$, $R(t, 0.5)$, $R(t, 0)$, $R(t, -0.5)$, $R(t, -0.8)$.

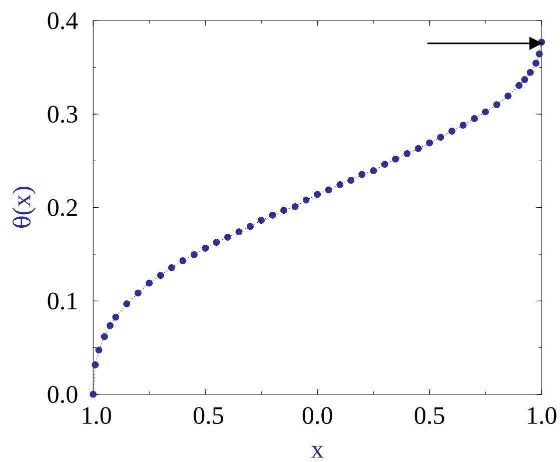

Figure 5. Exponent $\theta(x)$ for the one-dimensional Ising model. The arrow represents the usual persistence exponent $\theta = \frac{3}{8}$.

the Laplace transform of the autocorrelation function $A(\tau) = \langle\sigma(0)\sigma(\tau)\rangle$ with respect to the logarithmic time $\tau = \ln t$ (see equation (B.7)) [44].

The probability of persistent large deviations $R(t, x)$ decays algebraically, with an exponent continuously varying with x (figures 4 and 5). For $x = 1$ the usual persistence exponent $\theta = \frac{3}{8}$ is recovered.

5. The two-dimensional voter model

The voter model is defined as follows [42]. On each site of a d-dimensional lattice, opinions of a voter or values of a spin $\sigma = 1, 2, \ldots, q$ are initially distributed randomly. At each timestep a site is picked at random. The voter on this site takes one of the opinions of its $2d$ neighbours, with equal probabilities. Hence the rules of the dynamics of the voter model are a simple generalization of those of the one-dimensional Ising model at zero temperature. In particular the one-dimensional voter model is identical to the Glauber–Ising chain.

Earlier references to $P(t,x)$ or to the occupation time T_+ for the voter model may be found in [32, 34–36]. The function $a(t)$ appearing in the large deviation expression equation (2.4) is related to the variance of the occupation time T_+ [32, 34, 35], hence to the two-time correlation function of the process [44], by (see appendix A)

$$a(t) \sim \frac{t^2}{\operatorname{Var} T_+} \sim \frac{1}{\operatorname{Var} M}. \tag{5.1}$$

In one dimension, $\operatorname{Var} M$ = constant, as mentioned in section 4, hence $a(t) = \mathcal{O}(1)$. The convergence in distribution of T_+/t in one dimension was shown in [32]. In two dimensions, $\operatorname{Var} M \sim 1/\ln t$ [32, 34–36, 44], hence $a(t) \sim \ln t$. Therefore the rate at which the distribution of M becomes peaked is very slow. It was conjectured in [34, 35] that, for $d = 2$, the scaling hypothesis (A.7) is exact, and that therefore $P(t,x)$ should have algebraic decay at large times. For $d > 2$, $a(t)$ is respectively equal to $\sqrt{t}$, $t/\ln t$, t for $d = 3, 4$ and $d > 4$ [32, 34–36, 44].

We performed numerical simulations of the two-dimensional voter model, for system sizes up to $(4000)^2$, with $q = 2$ ($\sigma = \pm 1$). These simulations suggest that, in the regime of large deviations, $P(t,x)$ behaves as $t^{-\tilde{\theta}(x)}$, with an exponent continuously varying with x ($x > 0$), and to be identified to $I(x)$ defined in equation (2.4). Since $\operatorname{Var} M \sim 1/\ln t$, the regime of large deviations is very long to attain. We shall present the analysis of the scaling in [44]. We conjecture that $I(x) \to \infty$ when $x \to 1$.

The numerical results also seem to indicate that $R(t,x)$ behaves as $\exp[-J(x)(\ln t)^2]$, reminiscent of the behaviour of the usual persistence probability [16, 43]. Hence the function $b(t)$ introduced in (2.5) is equal to $(\ln t)^2$. It is nevertheless rather difficult to conclude, on the basis of our numerical simulations.

We conjecture that $b(t)$ is proportional to $N(t)$, the average number of particles in the dual particle system (diffusing coagulation, or equivalently reaction diffusion $\mathrm{A} + \mathrm{A} \to \mathrm{A}$, with a local source). $N(t)$ is equal respectively to $\ln t$, $(\ln t)^2$, $\sqrt{t}$, $t/\ln t$, t for $d = 1, 2, 3, 4$ and $d > 4$ [32, 34–36, 43]. These results have a clear intuitive interpretation. When the dimension of space increases, particles interact less strongly since they have increasingly more space to explore before meeting. As a consequence, they are less correlated and their average number increases. In a high enough dimension, the reaction between particles becomes irrelevant, hence $N(t)$ becomes proportional to t, reflecting the total independence of the particles. Note that above two dimensions, $a(t)$ is equal to $N(t)$ (and therefore to $b(t)$).

The voter model therefore interpolates between the case of the Glauber–Ising chain seen in previous section, if $d = 1$, and the case of the random walk of section 2, if $d > 4$.

6. Discussion and conclusion

The most striking conclusions that may be drawn from this study are, from our point of view, the existence of a limit law for the mean magnetization, and the appearance of families of exponents in the temporal decay of $R(t,x)$, the probability of persistent large deviations, both for the diffusion equation (in any dimension) and the one-dimensional Ising chain. More generally, these features are expected for any coarsening (or ageing) system for which temporal scaling takes place. Finally there are indications that the large deviations $P(t,x)$ for the two-dimensional voter model are algebraic. This extends the scope of former studies on the persistence exponents found in coarsening systems.

A number of comments are in order.

The time dependence of $a(t)$ and $b(t)$, defined in (2.4) and (2.5), is summarized in table 1. One observes that for the random walk, $a(t)$ and $b(t)$ are proportional to t, while these functions are slowly increasing with time for the coarsening systems considered in this work, or even constant. This may be interpreted as follows. In the case of the random walk, $M(t)$ is given by a sum of t independent random variables. In the coarsening systems studied here, values of the spin at a fixed position at different times are strongly correlated random variables. Thus $a(t)$ and $b(t)$ measure in some sense the effective number of independent variables in the system. This is clear for $a(t)$, from its very definition given in appendix A, and for $b(t)$, at least for the voter model, from the discussion given at the end of section 5.

Thus the mechanism by which $R(t, x)$—and as a consequence $R(t)$, the usual persistence probability—have algebraic decay at large times, for the diffusion equation and the one-dimensional Ising model, can be traced back to the logarithmic behaviour of $b(t)$ defined in equation (2.5). The same comment holds for $P(t, x)$ and $a(t)$ for the two-dimensional voter model.

Unfortunately the exact computation of the exponents seems a difficult task, since this amounts to computing the 'entropy' functions $I(x)$ or $J(x)$. Already computing the usual persistence exponent, corresponding in the present framework to taking the limit $x \to 1$, is in general difficult. Moreover, even for the simple random walk, the probability $R(t, x)$ is a nontrivial mathematical object [29]. At least for the diffusion equation, for which it is possible to obtain analytic results for $P(t, x)$ in the independent interval approximation (see section 3), one could hope of computing $R(t, x)$. Let us note that some aspects of this work, for instance the interpretation of the exponents as entropy functions, are reminiscent of the multifractal formalism.

One also observes that for coarsening systems, when $x \to 1$, $I(x)$ diverges, while $J(x)$ converges to a constant. The divergence of $I(x)$ signals the crossover from large deviations ($P(t, x)$) to persistence ($R(t)$). The convergence of $J(x)$ shows that $R(t, x)$ is a natural generalization of the persistence probability $R(t)$, $b(t)$ encoding the type of decay of the persistence probability, being algebraic or not.

We can enhance the difference between $R(t, x)$ and $P(t, x)$ as follows. First define the new random variable

$$\sigma(t, x) = \mathrm{sign}(M(t) - x) \tag{6.1}$$

which is an indicator of whether the mean magnetization $M(t)$ at time t is above or below the level x. One has

$$P(t, x) = \left\langle \frac{1 + \sigma(t, x)}{2} \right\rangle. \tag{6.2}$$

Then $R(t, x) = P(M(u) \geqslant x, \forall u \leqslant t)$ is just the persistence probability of this random variable. Therefore

$$R(t, x) = P(\sigma(u, x) = 1, \forall u \leqslant t) = \left\langle \frac{1 + \sigma(t_1, x)}{2} \frac{1 + \sigma(t_2, x)}{2} \cdots \frac{1 + \sigma(t_n, x)}{2} \right\rangle \tag{6.3}$$

(taking a discrete set of intermediate times, then letting $n \to \infty$) which shows that $R(t, x)$ is a highly nonlocal function of time.

One may generalize the present approach by progressively 'thinning' large deviations, and tracking increasingly rare events. Let us define

$$M(t, x) = \frac{1}{t} \int_0^t \mathrm{d}u\, \sigma(u, x) \tag{6.4}$$

($-1 \leqslant M(t,x) \leqslant 1$) and the corresponding probabilities $P(t,x,y) = P(M(t,x) \geqslant y)$ and $R(t,x,y) = P(M(u,x) \geqslant y, \forall u \leqslant t)$.

First consider the random walk. Take $x = 0$ for simplicity. Then $M(t,0)$ is simply related to the fraction of time the walker spends on the right-hand side of the origin. The limit distribution of this quantity (when $t \to \infty$) is given by the arc sine law [45].

We computed $R(t,x,y)$ on the diffusion equation, and on the Ising chain. For example, figure 2 shows $R(t,x,y)$, with $x = y = -0.8$, for the one-dimensional diffusion equation. Again algebraic decay is observed.

Let us point out that this progressive thinning of large deviations implies probing the system by events which are always more nonlocal in time. This questions the possibility of the existence of an infinite number of exponents in temporal quantities measured on strongly interacting systems.

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Appendix A

This appendix provides an explanation of the large deviation expressions (2.2) and (2.4).

A.1. Independent random variables

Consider $Y = tM = \sum_1^t \sigma(u)$, where the σ are independent identically distributed random variables. The generating function of moments of Y is

$$\langle \mathrm{e}^{sY} \rangle = \mathrm{e}^{tK(s)} \tag{A.1}$$

where $K(s) = \ln\langle \mathrm{e}^{s\sigma} \rangle$ is the generating function of cumulants of the random variable σ. An inverse Laplace transform yields

$$f_Y(y) = \int \frac{\mathrm{d}s}{2\pi\mathrm{i}} \mathrm{e}^{-sy+tK(s)} = \int \frac{\mathrm{d}s}{2\pi\mathrm{i}} \mathrm{e}^{-t[sx-K(s)]} \tag{A.2}$$

where $x = y/t$. For $t \to \infty$ we use the saddle-point method to evaluate the integral. At the saddle point, $K'(s_c) = x$. Defining

$$I(x) = s_c x - K(s_c) \tag{A.3}$$

yields $f_Y(y) \sim \mathrm{e}^{-tI(x)}$. Finally, at large times,

$$P(t,x) \sim \mathrm{e}^{-tI(x)} \tag{A.4}$$

which is (2.2). Note that $I(x)$ is the Legendre transform of $K(s)$.

Let us apply this general formalism to the case of the random walker (see section 2). Then $\langle \mathrm{e}^{s\sigma} \rangle = \cosh s$, and $K(s) = \ln\cosh s$. At the saddle point $x = K'(s_c) = \tanh s_c$, hence

$$s_c = \frac{1}{2} \ln \frac{1+x}{1-x}. \tag{A.5}$$

Noting that $\cosh s_c = (1-x^2)^{-1/2}$, leads to $I(x) = (\frac{1}{2})[(1+x)\ln(1+x) + (1-x)\ln(1-x)]$.

A.2. Correlated random variables

Consider the generating function of the cumulants of M, denoted by c_n,

$$K_M(s) = \ln\langle \mathrm{e}^{sM}\rangle = \sum \frac{c_n}{n!} s^n. \tag{A.6}$$

Assume that, when $t \to \infty$, the c_n scale as [33, 34]

$$c_n \approx \frac{b_n}{[a(t)]^{n-1}} \qquad (t \to \infty) \tag{A.7}$$

where $a(t)$ diverges with t, and the b_n are constants. Hence $a(t) \sim 1/\operatorname{Var} M$.

Now consider $Y = a(t)M$. Then

$$\frac{1}{a(t)} \ln\langle \mathrm{e}^{sY}\rangle = \frac{1}{a(t)} K_M(sa(t)) \approx \sum \frac{b_n}{n!} s^n \equiv K(s) \qquad (t \to \infty). \tag{A.8}$$

Hence at large times,

$$\langle \mathrm{e}^{sY}\rangle \approx \mathrm{e}^{a(t)K(s)} \tag{A.9}$$

from which one obtains

$$f_Y(y) = \int \frac{\mathrm{d}s}{2\pi \mathrm{i}} \mathrm{e}^{-sy+a(t)K(s)} = \int \frac{\mathrm{d}s}{2\pi \mathrm{i}} \mathrm{e}^{-a(t)[sx-K(s)]} \tag{A.10}$$

where $x = y/a(t)$. Continuing as above, we obtain, at large times,

$$P(t, x) \sim \mathrm{e}^{-a(t)I(x)} \tag{A.11}$$

which is (2.4). Again, $I(x)$ is given by (A.3) and $K'(s_c) = x$.

Here, the role of $a(t)$ parallels that played by t for the former case of independent variables. This function can be therefore interpreted as the effective number of independent variables, in the case where the spins σ at different times are correlated. This analysis applies to the voter model in dimension $d \geqslant 2$.

In the cases of the diffusion equation or the one-dimensional Ising–Glauber chain, $a(t) = \mathcal{O}(1)$, because all cumulants become constant when $t \to \infty$. Therefore $P(t, x)$ converges to a limit law $P_\infty(x)$ that we shall still write in the form (A.11), though it is not derived from (A.10). By extension, we shall still speak of large deviations when $t \to \infty$ and $x \approx 1$.

Appendix B

In this section we first show how to compute the moments of the distribution of the mean magnetization for the diffusion equation. We then solve the set of equations (3.13*a*–*c*) for the case of an exponential distribution of intervals, $f(l) = \theta \mathrm{e}^{-\theta l}$, proportional to the tail of the true distribution given by equation (3.6).

Using equations (3.13*a*–*c*), the moments $\langle M^k\rangle$ can be computed recursively as follows. The computation is completed in three steps.

(i) From (3.13*c*) one computes the moments of X from those of e^{-l}, i.e. as functions of the coefficients $\hat{f}_k$, recursively, where $\hat{f}_k$ denotes $\hat{f}(s)$ for integer values of the argument.

For instance

$$\begin{aligned}
\langle X\rangle &= \frac{1}{1+\hat{f}_1}\\
\langle X^2\rangle &= \frac{1-\hat{f}_1}{(1+\hat{f}_1)(1-\hat{f}_2)}\\
\langle X^3\rangle &= \frac{1-2\hat{f}_1+2\hat{f}_2-\hat{f}_1\hat{f}_2}{(1+\hat{f}_1)(1-\hat{f}_2)(1+\hat{f}_3)}\\
&\dots.
\end{aligned} \tag{B.1}$$

(ii) From (3.13b), and using equation (3.10), one has

$$\langle \xi^k\rangle = \langle \mathrm{e}^{-\lambda k}\rangle\langle X^k\rangle = \frac{1-\hat{f}_k}{k\langle l\rangle}\langle X^k\rangle. \tag{B.2}$$

(iii) By symmetry, only even moments of M are nonzero. From (3.13a) they are related to those of ξ by a binomial expansion.

Finally one obtains

$$\begin{aligned}
\langle M^2\rangle &= 1-\frac{2}{\langle l\rangle}\frac{1-\hat{f}_1}{1+\hat{f}_1}\\
\langle M^4\rangle &= 1-\frac{8}{3\langle l\rangle}\frac{(1-2\hat{f}_1+2\hat{f}_2-\hat{f}_1\hat{f}_2)(1-\hat{f}_3)}{(1+\hat{f}_1)(1-\hat{f}_2)(1+\hat{f}_3)}\\
&\dots.
\end{aligned} \tag{B.3}$$

Replacing $\hat{f}(s)$ by its expression in terms of $\hat{A}(s)$ (see equation (3.6)), permits us to recast (B.3) into

$$\begin{aligned}
\langle M^2\rangle &= \hat{A}_1\\
\langle M^4\rangle &= 1-\frac{(1-3\hat{A}_1+4\hat{A}_2)(1-3\hat{A}_3)}{1-2\hat{A}_2}.
\end{aligned} \tag{B.4}$$

The first line of equation (B.4) may be understood as follows. In the long-time regime, using the logarithmic time $\tau = \ln t$, one has

$$\langle M^2(\tau)\rangle = 2\mathrm{e}^{-2\tau}\int_0^{\tau} \mathrm{d}\tau_2\, \mathrm{e}^{\tau_2}\int_0^{\tau_2} \mathrm{d}\tau_1\, \mathrm{e}^{\tau_1} A(\tau_2-\tau_1) \tag{B.5}$$

where $A(\tau)$ is the autocorrelation function (3.2). Laplace transforming both sides of (B.5) gives

$$\int_0^{\infty} \mathrm{d}\tau\, \mathrm{e}^{-s\tau}\langle M^2(\tau)\rangle = \frac{2\hat{A}(s+1)}{s(s+2)} \tag{B.6}$$

hence, when $\tau\to\infty$,

$$\langle M^2\rangle = \hat{A}(1) \tag{B.7}$$

since the rightmost pole of the right-hand side of (B.6) is at $s = 0$. The result (B.7) is generic for coarsening systems, whenever the autocorrelation function is scaling in the two time variables. In particular it holds for the Ising chain studied in section 4.

We now show that the solution of equations (3.13*a*–*c*) for $f(l) = \theta e^{-\theta l}$ is the beta law (3.18). Setting $Z = e^{-l}$ in (3.13*c*), the integral equation for the invariant distribution f_X reads

$$f_X(x) = \int_0^1 dz \int_0^1 dy\, f_Z(z) f_X(y) \delta(x - 1 + zy) \tag{B.8}$$

where f_Z, the probability density function of the variable Z, is known from that of the interval length l, $f(l)$. For $f(l) = \theta e^{-\theta l}$ one has $f_Z(z) = \theta z^{\theta - 1}$, which when cast into equation (B.8) leads to the solution

$$f_X(x) = B^{-1}(\theta + 1, \theta) x^{\theta} (1 - x)^{\theta - 1} \tag{B.9}$$

where $B(\theta + 1, \theta)$ is the beta function. Then computing

$$\langle \xi^s \rangle = \langle e^{-\lambda s} \rangle \langle X^s \rangle \tag{B.10}$$

with $\langle e^{-\lambda s} \rangle = \hat{q}(s) = \theta/(s + \theta)$, and $\langle X^s \rangle = B^{-1}(\theta + 1, \theta) B(\theta + s + 1, \theta)$ leads to $\langle \xi^s \rangle = B^{-1}(\theta, \theta + 1) B(\theta + s, \theta + 1)$ hence to the law

$$f_\xi(x) = B^{-1}(\theta, \theta + 1) x^{\theta - 1} (1 - x)^{\theta} \tag{B.11}$$

for the random variable ξ. Finally, for this choice of $f(l)$, the solution of (3.13*a*–*c*) is a beta law on $(-1, 1)$

$$f^{\mathrm{Beta}}(x) = B^{-1}(\tfrac{1}{2}, \theta)(1 - x^2)^{\theta - 1} \tag{B.12}$$

which is equation (3.18).

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